GRAPH

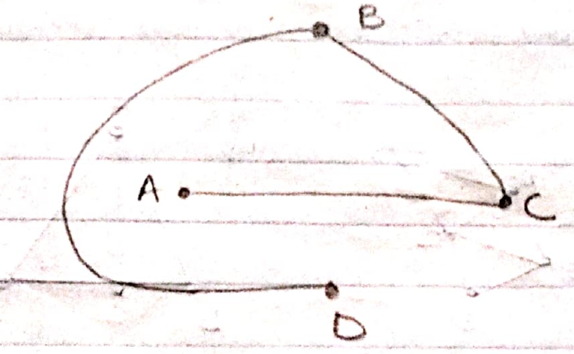
Definition (Graph): Let V be a finite non-empty set. Let E be a finite set such that each element of E represent an unordered pair of elements in V. Then $G = (V, E)$ is called a graph.

Each element of the set V is called a Vertex and each element of the set E is called an Edge.

Example:

Suppose $V = \{A, B, C, D\}$ and $E = \{(A,C), (B,C), (B,D)\}$. The element in the set E can also be labelled as $e_1 = (A, C)$, $e_2 = (B, C)$, $e_3 = (B, D)$.

The graph of $G = (V, E)$ is represented below



The set E can be empty when this happens G is said to be a null graph.

Example of Null graph

Definition (End Vertices):

If the edge e of a graph is given by $e = (A, B)$ then the vertices A and B are called end vertices of the edge e. Recall

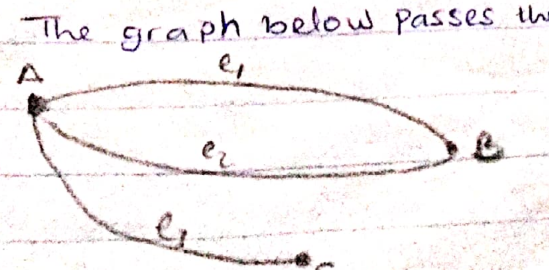
Recall that: Similarly A can be a starting vertex or a final vertex i.e

i.e $e = (A, B) = (B, A)$ because of the unorderded pair of vertices A and B.

Definition ⟨Parallel Edges⟩:

Parallel Edges are edges whose end vertices are the same.

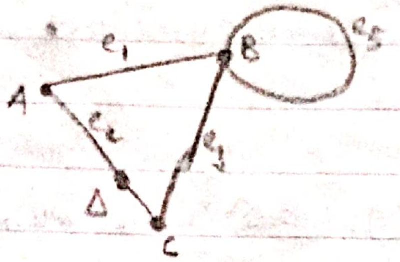
Example: The graph below passes the parallel edges e_1 and e_2



Definition <LOOP>:

An edge in a graph is called a loop if the starting and final vertices are the same.

e.g: The edge e_5 is a loop



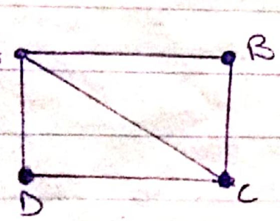
$e_5 = (B,B)$ is a loop

Definition <Simple Graph>.

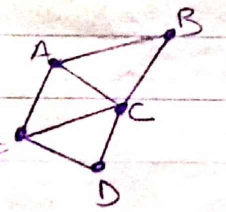
A graph which does not contain any parallel edges or loops is called a simple graph

e.g

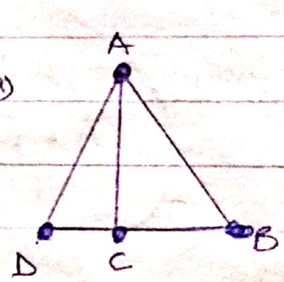
(i)



(ii)



(iii)

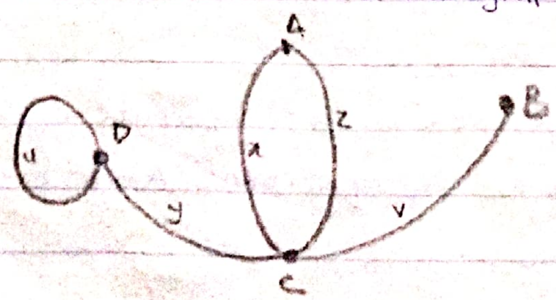


Definition <Multigraph>

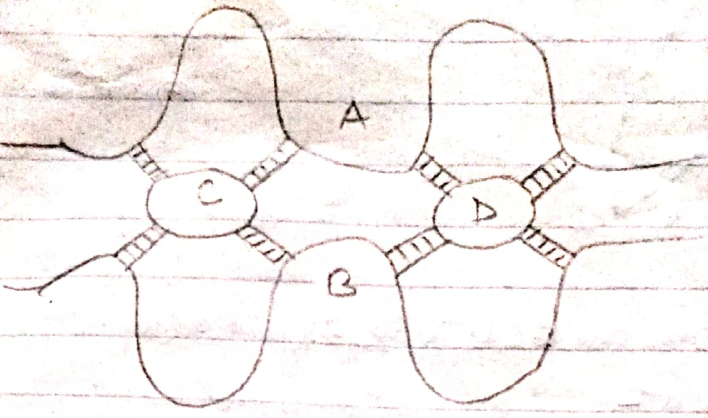
A graph that contains atleast one loop or a parallel edges is called a multigraph

e.g

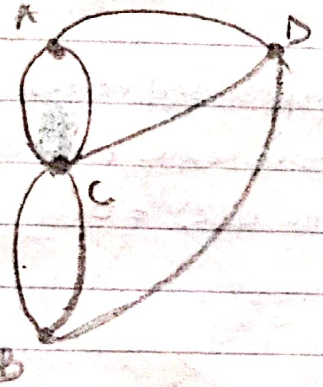
$V = \{A, B, C, D\}$ $E = \{x, y, z, u, v\}$, where $x = (A,C)$ $y = (C,D)$, $z = (A,C)$, $u = (D,D)$ $v = (B,C)$. $G = (V,E)$ is a multigraph represented pictorially as follows.



KONIGSBERG BRIDGE PROBLEM



The Konigsberg bridge problem can graphically be represented pictorially as.



even vertices that is two deg common

The konigsberg bridge problem was solved by labelling the islands as vertices and each bridge was represented by an edge. It was discovered that each islands were connected with odd numbers of bridges since there are odd numbers of edges incident to each vertex

Definition <NULL PARALLEL EDGE>

Two null parallel edges are said to be adjacent if they are incident with a common vertex

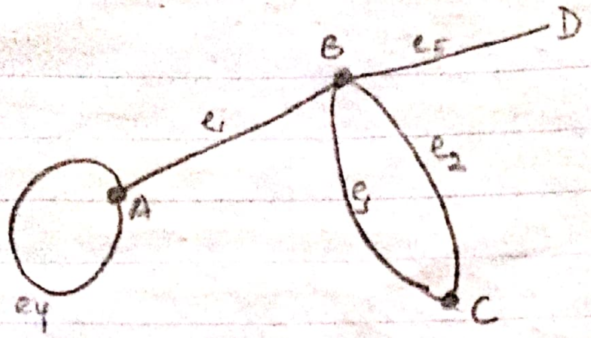
Definition

Let G be a graph and V be a vertex in G. Then the total numbers of edges which are incident with V is called the degree of V denoted by d(v)

Example;

Considering the konigsberg bridge problem graph the degree of vertex C is given by d(C) = 5 and degree of vertex A is given by d(A) = 3.

The degree of vertex d(v) is 1



The degree of vertex d(D) is 1

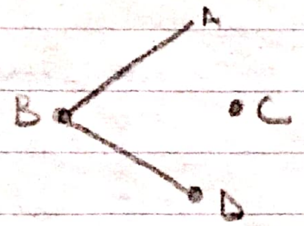
The vertex A is incident with one (1) loop e_4 and edge e_1

Therefore the degree of d(A) is 2

Definition

A vertex whose degree is zero (0) is called an isolated vertex

e.g:



C is an isolated vertex bcos there's no edge from vertex C to any other vertex

Therefore the d(C) = 0

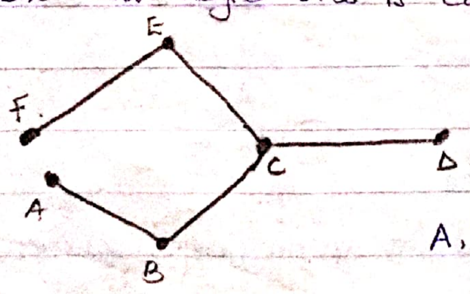
Definition <Bridge>

An edge in a graph is called a bridge if ~~the~~ deleting the edge left the graph disconnected

Pendant Vertex

A vertex with degree one is called ~~D~~pendant vertex.

e.g:



A, D and F are pendant vertices

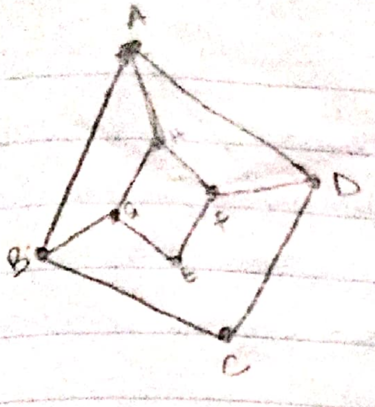
Theorem 1

Let $G = (V, E)$ be a graph then $\sum_{v \in V} d(v) = n(E)$ where n(E) implies number of edges

Defn <Even and ODD vertex>

If the d(v) is even, then v is said to be even. However if d(v) is odd, then v is said to be odd vertex e.g.?

(A)



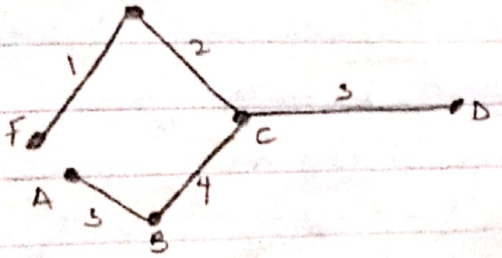
Theorem 2

The number of odd vertices in d graph is always even. Consider the example (A) above.
The odd vertices are 6

SERIES OF EDGES

Two adjacent edges (Adjacent edges are not parallel) are said to be in series if the degree of the common vertex is two.

e.g. Consider the graph



1 and 2 are in series

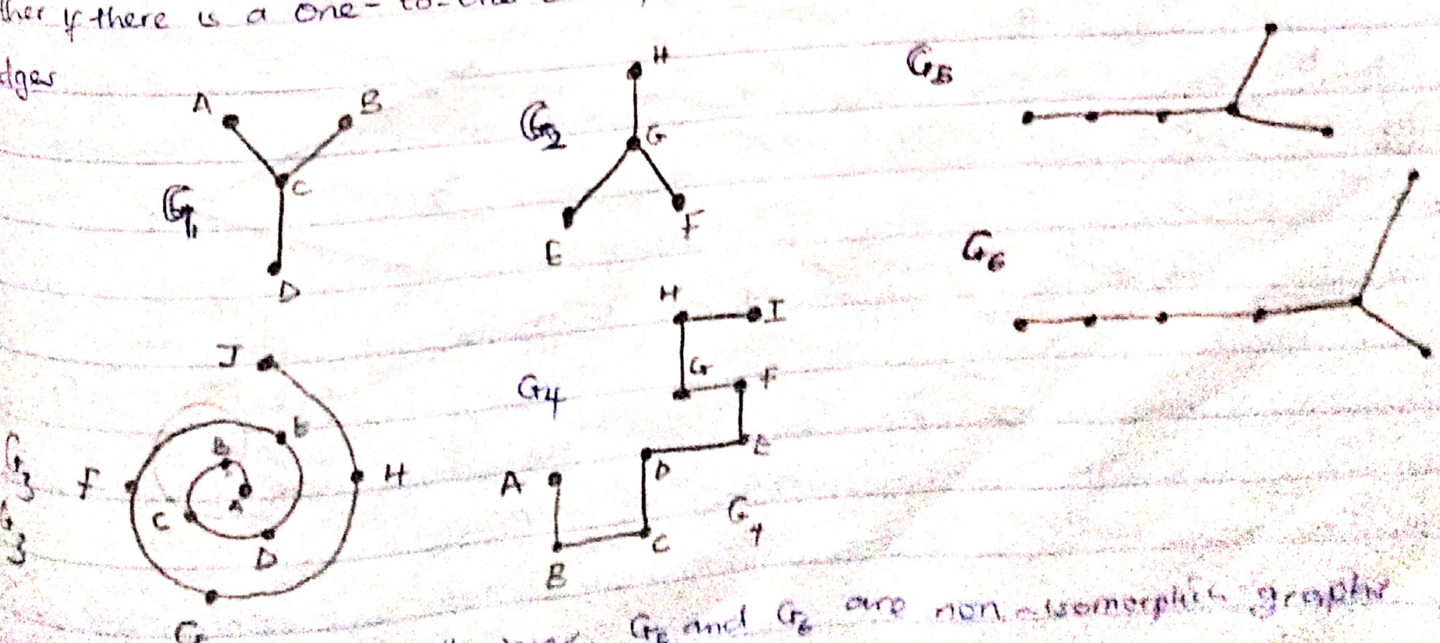
2 and 3 are not series

3 and 4 are not series

4 and 5 are in series

2 and 4 are not in series because the degree of vertex C connecting edge 2 and edge 4 is all the degree

ISOMORPHIC GRAPH: Two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and between their edges



However, G_5 and G_6 are non-isomorphic graphs

REMARK:

If two graphs are isomorphic then they must have:

1 The same number of vertices

2 The same number of edges

3 An equal number of vertices with a given degree. But the converse is not true.

SUBGRAPH

Defn: A graph $G_2 = (V_2, E_2)$ is said to be a subgraph of $G_1 = (V_1, E_1)$ if

1 $V_2 \subseteq V_1$

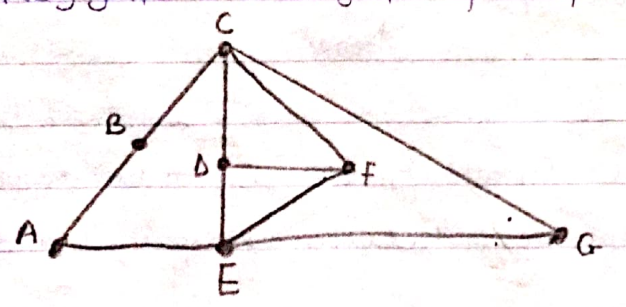
2 $E_2 \subseteq E_1$

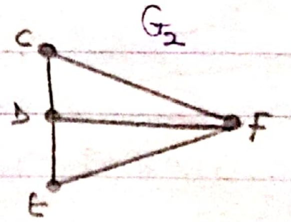
3 Each edge in G_2 has the same end vertices as in G_1

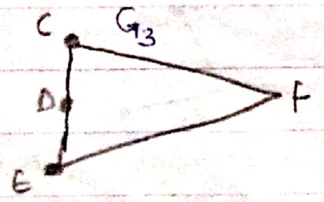
G_2 is a subgraph of G_1 is symbolically represented as

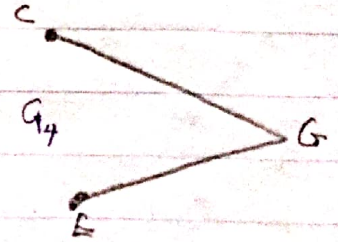
$$G_2 \leq G_1$$

Note that every graph is a subgraph of itself.









$G_2 \leq G_1$ $\quad G_3 \leq G_2$ $\quad G_4 \leq G_1$ $\quad G_3 \leq G_1$

Observe clearly that to remove a vertex is to remove the vertex together with the edges, incident with the vertex.

Also to remove an edge, is to remove the edge only and nothing more.

REMARK:

1. Every graph is it's own subgraph
2. A subgraph of a given subgraph of G is a subgraph of G
3. A single vertex of G is a subgraph of G
4. A single edge together with it's end vertices is a subgraph of G.

Defn <EDGE DISJOINT SUBGRAPH>:

Let G_1 and G_2 be two subgraph of G and let $E_1 \cap E_2 = \emptyset$ then G_1 and G_2 are said to be edge disjoint subgraph.

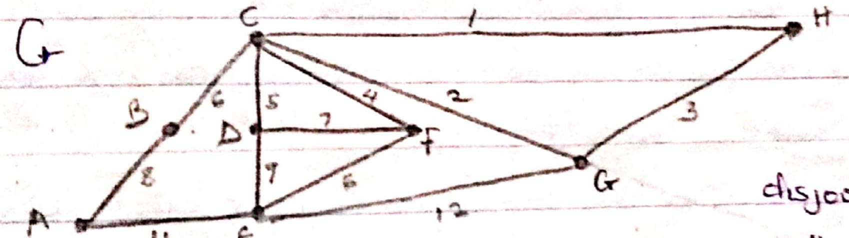
Defn <VERTEX DISJOINT SUBGRAPHS>

Let G_1 and G_2 be two subgraphs of G such that $V_1 \cap V_2 = \emptyset$ then G_1 and G_2 are called vertex disjoint subgraphs of G.

Note that: vertex disjoint $\Rightarrow$ edge disjoint

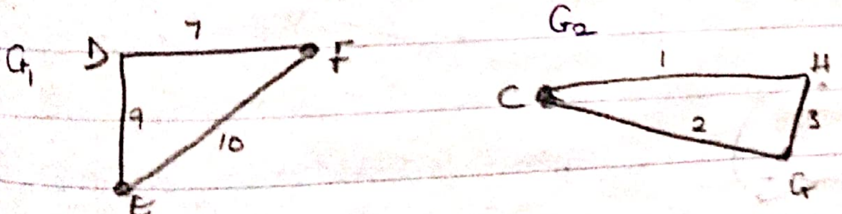
Edge disjoint $\not\Rightarrow$ Vertex disjoint

Examples

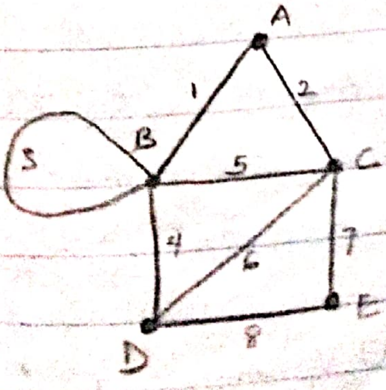


G_1 and G_2 are vertex disjoint subgraphs of G.

However, G_1 and G_2 are isomorphic subgraphs



Furthermore, G_1 and G_2 are Edge disjoint subgraphs of G.



WALK

Consider the following sequence

A, 2, C, 6, D, 8, E

This sequence is called a walk

Defn <WALK or EDGE-TRAIN>

A finite alternating sequence of vertices and edges <Begining and ending with vertex> such that each edge is incident with the vertices preceeding and following and such that no edge appears more than once is called a WALK or Edge Train or CHAIN

w_1: C, 2, A, 1, B, 3, B, 4, D, 6, C, 7, E, 8, D } are called walk

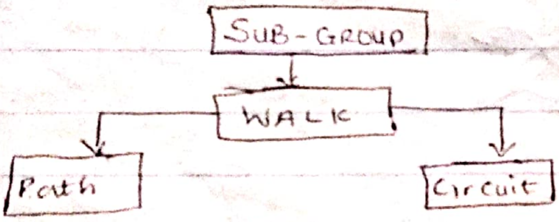
w_2: B, 5, C, 7, E, 8, D, 4, B

Defn: A walk in which a starting vertex is different from final vertex is called Open Walk

Defn: A Walk that is not open is called a closed Walk

Defn <PATH>: A Open walk in which one vertex appears more than one is called a path

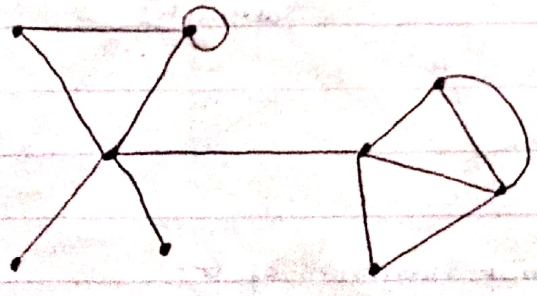
Defn <Length of the Path>: The number of edges in a path is called the length of the path.



Defn <Circuit>: A closed walk in which no vertex more than one, except the initial and final vertex is called a circuit

Defn: A Subgroups of G is said to be connected if there exist atleast one path between any two vertex of G.

Examples:



Defn <Components>; Let G be a graph and let v be a vertex of G. Then vertex V and all other vertices of G that have path to V together with all the edges incident to them form a Components.

Theorem 3; A graph G is disconnected if $V = V_1 \cup V_2$ where $V_1, V \neq \phi$ and there exist $x_0 \in V_1$ and $Y_0 \in V_2$ and there is no edge between x_0 and Y_0 as ends vertices

Theorem 4:- If a graph G has exactly two odd vertices, then their is a path connecting them.

Proof:- Let x and y be the two odd vertices of G. Consider the component of G that contains x. If this component does not contain y, then the components where the graph contains x only is an odd vertex which is a contradiction by theorem 2. Therefore, the component containing x also contain y implying that there is a path from x to y.

Defn <Simple Graph>: A simple graph is a graph which have no parallel edges and self loops.

Lemma 1:- The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

* Theorem 5: Let G be a simple graph with k- components and n vertices, then G can have at most $\frac{(n-k)(n-k+1)}{2}$ number of edges.

Proof:

Suppose the i^{th} components contains n_i vertices. It means that

$$\sum_{i=1}^{n} n_i = n_1 + n_2 + \dots + n_k = n$$

Now for $n_i \geq 1$ for $i = 1, 2, \dots, k$

$$(n_i - 1) \geq 0 \text{ for } i = 1, 2, \dots k$$

$$\sum (n_i - 1) \leq n - k$$

Squaring both side we have

$$\left(\sum (n_i - 1)\right)^2 = (n-k)^2$$

$$\sum (n_i^2 - 2n_i + 1) \leq (n-k)^2$$

Taking the individual summation

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq (n-k)^2$$

$$\sum n_i^2 - 2n + k \leq (n-k)^2$$

we $\sum n_i^2 - 2n + k \leq (n-k)^2$

i.e. $\sum n_i^2 \leq (n-k)^2 + 2n - k$

$$\sum n_i^2 \leq (n^2 - k)(2n-k) + (2n-k)$$

$$\sum n_i^2 \leq n^2 - (2n-k)(k-1) \quad \text{—— (A)}$$

Now the maximum number of edges in the i^{th} component is $\frac{n_i(n_i - 1)}{2}$

Therefore the maximum number of edges in G is half

$$= \tfrac{1}{2}\left(\sum n_i^2 - n\right) \leq \tfrac{1}{2}(n-k)^2 + 2n - k - n),$$

$$\frac{1}{2}\sum n_i(n_i - 1) = \frac{1}{2}\left(\sum n_i^2 - \sum n_i\right)$$

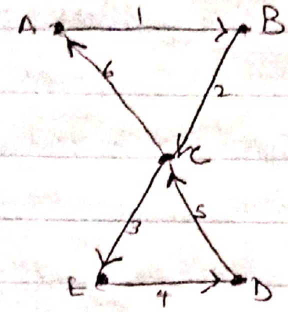
From eqn (a)

$= \frac{1}{2}(n-k)(n-k+1)$

As required proof.

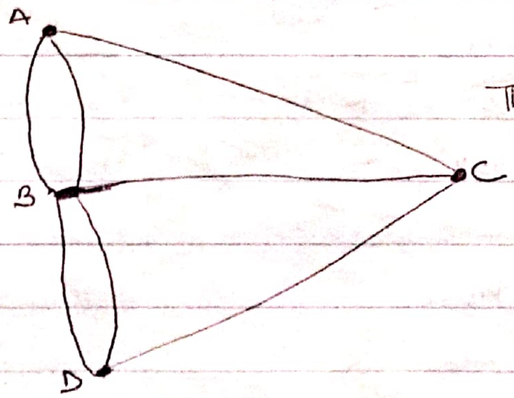
Defn: <Euler Line>: A closed walk running through all edges of G exactly once is called Euler line.

E.g.:



A B C E D C A (1, 2, 3, 4, 5, 6)

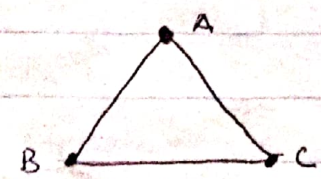
Defn <Euler Graph>: A graph in which containing Euler line is present is called Euler graph



This graph is not Euler

The following theorem enable us to determine Euler graph

Theorem 6: A connected graph G is Euler iff all it's vertices are even.

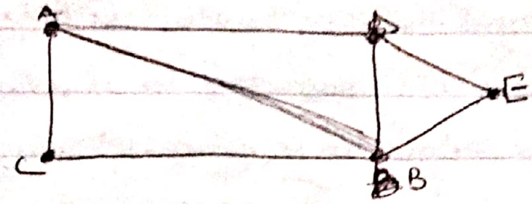


Euler graph

The Star of David is an Euler graph

Furthmore, Muhammad Scemitars is an Euler graph

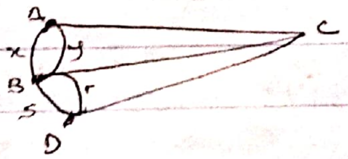
However, below is an UnEuler graph



Exactly 3 of the vertices are odd.

Suppose a connected graph G has exactly two odd vertices X and Y, there exists a connected line from X to Y

<u>Theorem 7</u>: Let G be a connected graph with 2k odd vertices then, there exist k edge disjoint subgraphs of G. Such that each is unicursal and they together contain all edges of G.



Unicursal Graph

A walk in which every edge is covered exactly once and the starting vertex and the ending vertex are different is called <u>UNICURSAL GRAPH</u>.

OPERATIONS ON GRAPHS

Definition: Let $G=(V,E)$, $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ then $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2$
$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

Ring Sum of Graphs

$G_1 \oplus G_2$ is the graph that contains all the vertices in G_1 and G_2 and of edges in G_1 or G_2 but not both G_1 and G_2.

idop
identpotent

REMARKS

If the edges that are common in G_1 and G_2 are removed from $G_1 \cup G_2$, Then the graph left is the ring sum $G_1 \oplus G_2$.

Some Observations:

i. The operations of Union, Intersection and ring sums of graphs are commutative
ii. If G_1 and G_2 are edge disjoint graphs then $G_1 \cap G_2$ is a null graph or empty graph
iii. If G_1 and G_2 are Vertex disjoint graphs then $G_1 \cap G_2$ is empty
iv. For any graph $G \cup G = G \cap G = G$ and the ring sum of $G \oplus G$ is a null graph.
v. If G_1 is a subgraph of G then, the ring sum $G \oplus G_1$ is also a Subgraph of G. This is because after removing the edges in G_1 from the ring sum edge set the resultant set is still a subset of G edge set.
vi. $G - G_1$ is the complement of Graph G_1 in G

Definition

A graph G is said to be decomposed into two subgraphs G_1 and G_2 if $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \phi$ or a null graph i.e Every edge in the Graph G is either in G_1 or in G_2 but not both in G_1 and G_2. It can be seen very easily that a graph containing m-edges can be decomposed into $2^{m-1} - 1$ different ways into two subgraphs

Definition

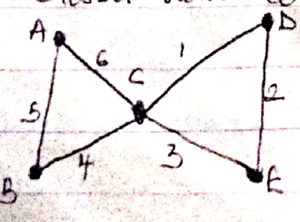
Suppose G is a graph and v is a vertex in G, then $G - v$ is a graph left after the removal of vertex v in G.

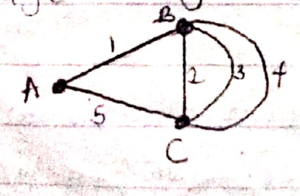
Definition

If G is a Graph and e is an edge in G, then $G - e$ is a graph left after the removal of edge e in G.

* Defn: <Euler Graph>: ~~Closed walk~~

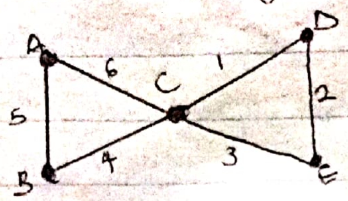
Closed walk containing each edge exactly Once





Theorem 8 ÷ ~~A G~~

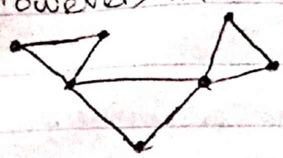
A connected graph G is euler iff each vertex of G is even.



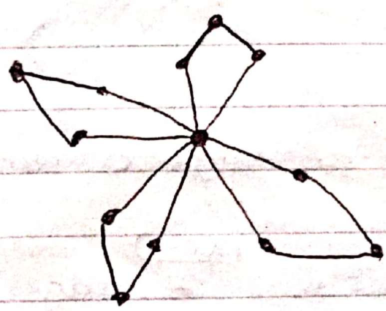
C1D2E3C is arbitrarily traceable from C

Also, C4B5A6C

However, this euler graph below ~~is~~ isn't traceable from a single vertex



~~This~~ Futhermore, the graph ~~above~~ below is abitrarily traceable from a single vertex



Theorem 9 ÷

A connected graph G is euler iff it can be decomposed into circuits

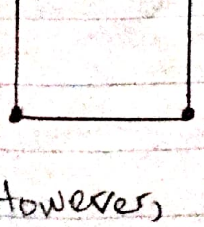
Theorem 10 ÷

An euler graph G is arbitrarily traceable from a vertex v iff every circuit in G contains the vertex v

Defn < Hamiltonian Circuit >

A closed walk tranverses every vertex in a graph G ~~excel~~ exactly once (except the starting and the ending vertex) is called the Hamiltonian circuit.

Examples



This graph is Euler and Hamiltonian

However,

is Euler but not Hamiltonian

This is also a Hamiltonian Circuit

Sir William Rowan Hamilton <1859>

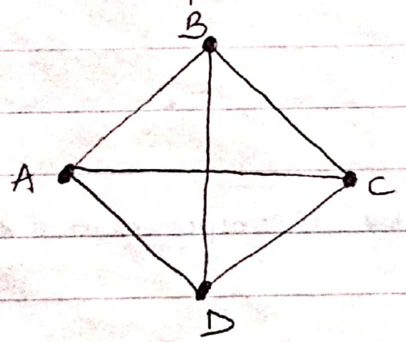
Hamiltonian path has the starting vertex different from the final vertex

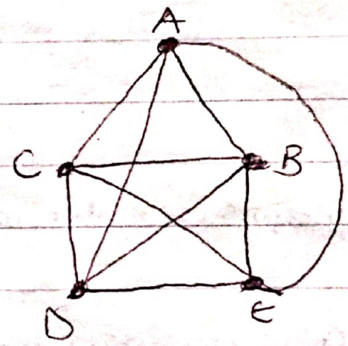
If there are n-vertices in a graph G and G contains a Hamiltonian path ~~and~~ then the length of the path is $n-1$.

Defn<Complete Graph>

A ~~Single~~ Simple graph in which there exist an edge between any pair of vertices is called a complete graph.

The number of edges in a graph of n-vertices is $\frac{n(n-1)}{2}$ edges.



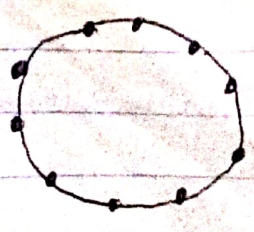


Theorem 11:

Let G be a Complete graph with n-vertices, n being odd number greater or equal to 3. Then there exist $\frac{(n-1)}{2}$ edge disjoint Hamiltonia circuits.

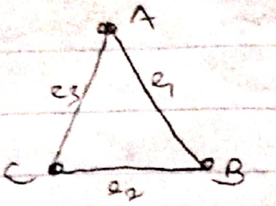
Theorem 12:

If G is a Simple graph with n-vertices and the degree of each vertex is atleast $n/2$ then G will have a Hamiltonia Circuit



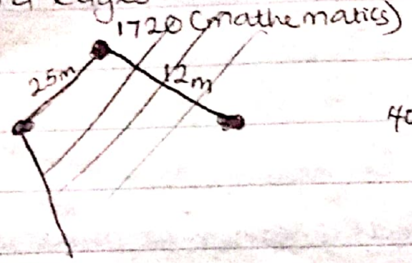
Defn <Labelled Graph>

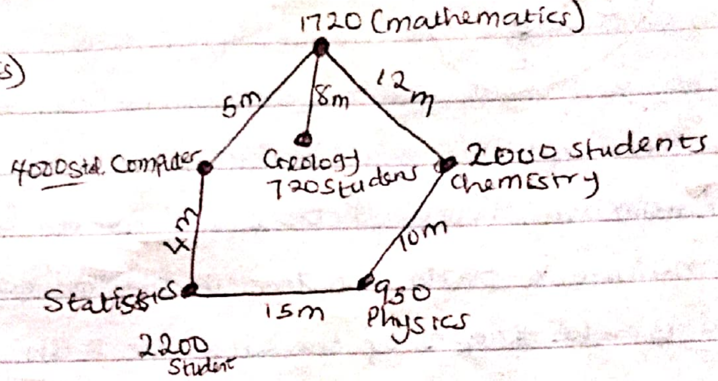
A labelled graph is a graph on which the edges and vertices are attached with variables or name



Defn <Weighted Graph>

A weighted graph is a graph that contains data attached to each of its ~~vertices and each~~ vertices and edges.

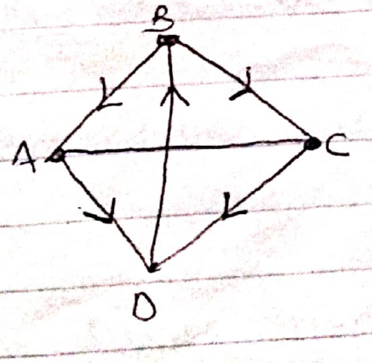




Another real-life application of a weighted graph is the map of Nigeria indicating the population of each cities as vertices, while the roads connecting each cities as edge is weighted as distance

* Defn < Directed Graph> or <Digraph>

A connected graph G in which the edges are ~~labelled~~ depicted with arrows indicating direction of flow of connection between the pairs of vertices is called a directed Graph.



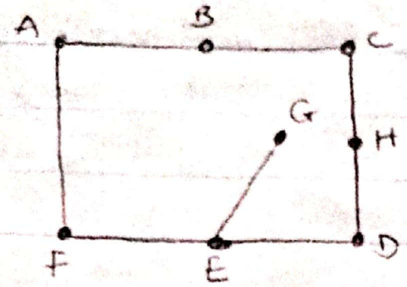
B, C, D, B, A, D, B

Defn < ~~Matrix~~ Distance >

let u and v be any two vertices in a connected graph G, the distance from u to v is the shortest path or minimum path between u and v.

This is denoted symbolically as $d(u,v)$.

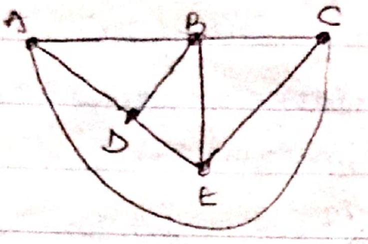
Example: Consider the Graph.



$\alpha = A, B, C, H, D = 4$

$\beta = A, F, E, D = 3$

$d(A, D) = 3$



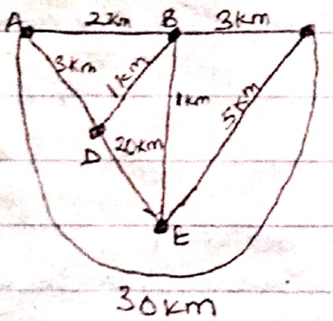
$d(A, E) = 2$

Minimum Path Problem in a Graph

The minimum path problem in a connected graph G that is labelled and weighted is the length of the shortest path btw any pair of vertices

Example

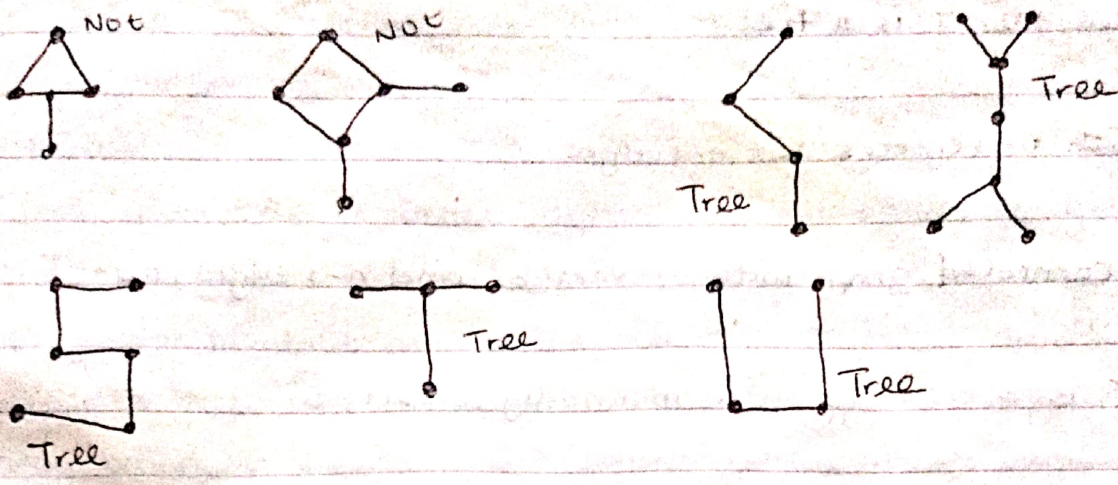
Consider the weighted graph below, what is the distance between A and C



$d(A, C) = 5km$

TREE

A tree is a connected graph without any circuit



A connected graph G can be made into a tree through suitable remova of edges to turn the graph circuitless but remain connected.

~~A vertex x in a graph G is called a cut~~

Defn <Cut-point>

A vertex v in a graph G is called a cut-point if the subgraph G-v left the graph G disconnected.

Theorem 13 ÷ A connected graph G is a tree iff there's one and only one path between any two vertices of G

Theorem 14 ÷

Let T be a connected graph with n-vertices such that ~~of T is circuitless~~

a T is circuitless

b There is one and only one path btw any vertices T

c T has n-1 edges

d T is minimally connected, then T is a tree

Definition

A graph G is said to be minimally connected, if removal of any edge G makes G disconnected

Theorem 15 ÷

There's one and only one path between any two vertices of a tree T

Theorem 16:-

Let T be a graph such that there's one and only one path between any pair of vertices, then T is a tree

Theorem 17:-

A Tree with n-vertices has n-1 edges

Theorem 18:-

If T is a connected graph with n-vertices and n-1 edges, then T is a tree

Theorem 19:-

A graph T is a tree iff it is minimally connected

Theorem 20:-

If T is a graph with n-vertices, n-1 edges and circuitless then T is connected

Theorem 21:-

Let T be a graph with n-vertices, then the following statements are equivalent

1 T is connected and circuitless

2 T is connected and has n-1 edges

3 T is circuitless and has n-1 edges

4 There's exactly one path between every pair of vertices in T

5 T is minimally connected

Defn <PENDANT VERTEX>

A vertex with degree 1 in a graph G is called a Pendant Vertex

* Theorem 22

If T is a tree with atleast two vertices, then T must have atleast 1 pendant vertex

Proof: <By Contradiction>

Let on contrary T has no pendant vertex, it means that the degree of each vertex is either two or more. Let T has n-vertices say $x_1, x_2, ---, x_n$

$\sum_{i=1}^{n} d(x_i) \geq \sum_{i=1}^{n} 2$ i.e $\sum_{i=1}^{n} d(x_i) \geq 2n$

By theorem 1

≥ 2 <Total number of edges in T>

Since T is a Tree it must have n-1 edges. Therefore we have

$2(n-1) \geq 2n$

$\Rightarrow -2 \geq 0$ which is impossible

So our ~~Supp~~ Sup-Proposition ~~that~~ so our assumption of pendant vertex is false. Hence T has atleast one pendant vertex.

Theorem 23:

A Tree with two or more vertices has atleast two pendant vertices

Proof:

Suppose one pendant vertex is there, then $\sum d(x) \geq 1 + \sum_{i=1}^{n-1} 2$

Since T is a tree with ~~having~~ $n-1$ edges, we must have $2(n-1) \geq 1 + 2(n-1)$.

$\Rightarrow 0 \geq 1$ which is false. Which contradicts our previous statement that has one pendant vertex

Hence it must has atleast two pendant vertex.

Defn (Metric)

Let S be a non-empty set. Let $d(x,y)$ be defined for each $x, y \in S$. Then $d(x,y)$ is called a distance between x and y if the following axioms are satisfied

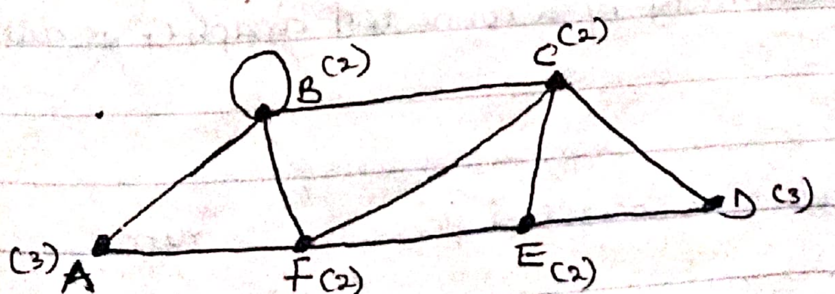
1. $d(x,y) \geq 0$ for every $x, y \in S$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x)$, $\forall x, y \in S$
4. $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x, y, z \in S$

Thus d is a metric on S and the pair (S,d) is called Metric Space

Defn

Let G be a connected graph and let (x,y) be any two vertices of G. Let the distance between x and y be defined as follows.

$d(x,y)$ = The length of the shortest path from x to y

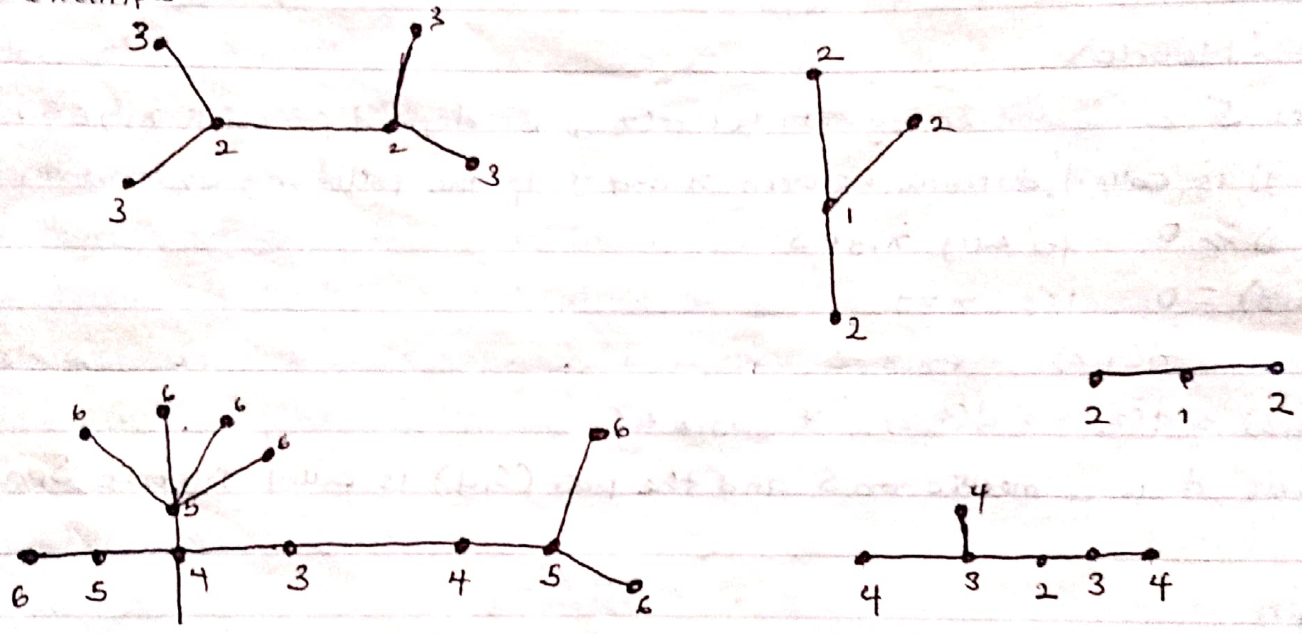


d	A	B	C	D	E	F
A	0	1	2	3	2	1
B	1	0	1	2	2	1
C	2	1	0	1	1	1
D	3	2	1	0	1	2
E	2	2	1	1	0	1
F	1	1	1	2	1	0

Defn <ECCENTRICITY>

Let G be a connected graph and let x be a vertex of G, then the eccentricity of x, denoted as $E(x)$, is defined as $E(x) = \max d(x,v)$ $v \in V$

For example



Definition <Center of G>

A vertex with minimum eccentricity of a connected graph G is called a Center of G

Theorem 24:-

The distance between vertices of a connected graph is a metric

Proof:-

Let x and y be any two vertices of G

Let B = The set of lengths of all paths from x to y

Let A = The set of lengths of all paths from x to y through z

It follows that $A \subseteq B$

Therefore $\inf(A) \geq \inf(B)$ as such $d(x,y) + d(y,z) \geq d(x,z)$

Theorem 257

Every tree has either ~~One~~ one or two centers.

REMARK:

If a Tree has two Center's then they must be adjacent.

Defn <RADIUS>

The eccentricity of a center of a tree is called the Radius of the tree

Definition <DIAMETER>

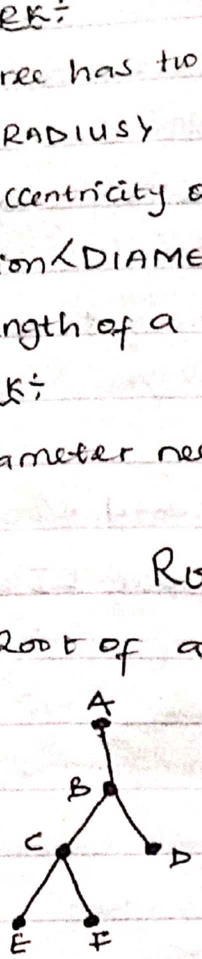
The length of a longest path in a tree is called the Diameter of the Tree

REMARK:

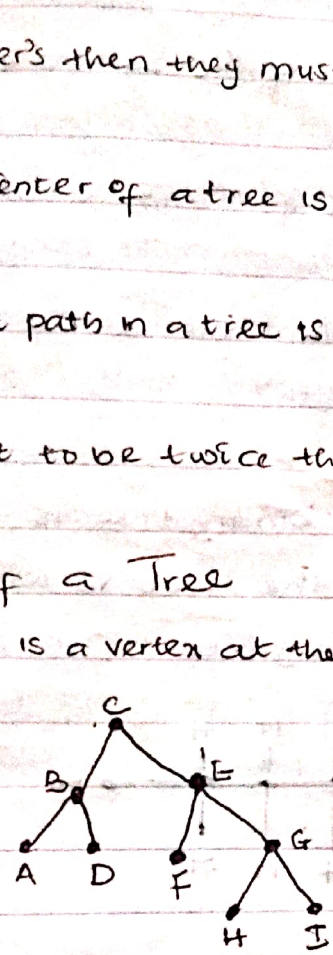
The Diameter need not to be twice the radius of a tree

Root of a Tree

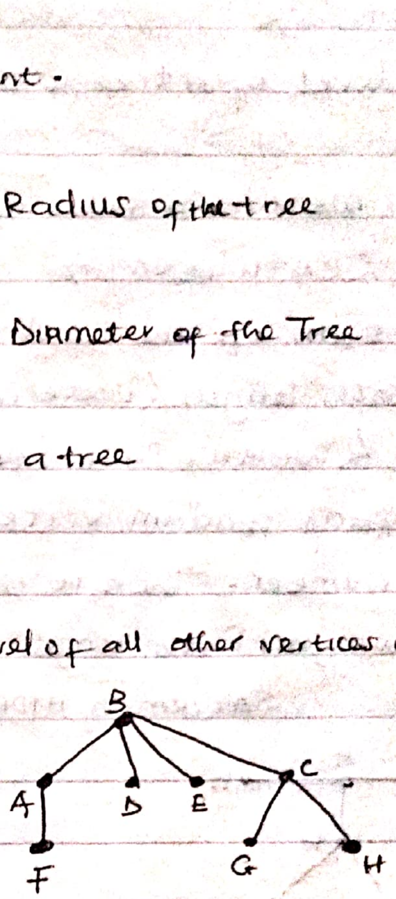
Defn: Root of a tree is a vertex at the heighest level of all other vertices of the tree.



Root is A

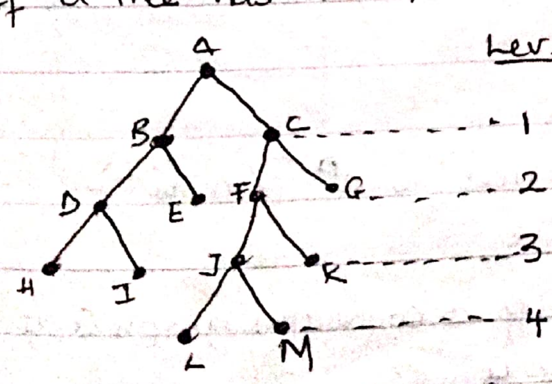


Root is C



Root is B.

If a tree has a root, then it is called a rooted tree



Level	Vertices
1	B, C
2	D, E, F, G
3	H, I, J, K
4	L, M

Defn <Binary Tree>

A Binary tree is a rooted tree having degree (2) while the remaining vertices is either of degree (3) or (1).

The diagram above illustrate a typical rooted tree.

Theorem 26:

· If t is a binary tree with n-vertices then:

1. n must be odd
2. T must have $\frac{n+1}{2}$ pendant vertices

Defn

The level of a tree is the distance of the root to each vertices

Defn

The maximum value of the level of a tree is called the height of the tree

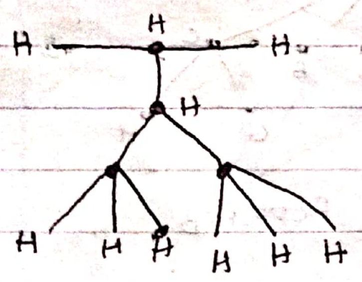
Defn <PATH LENGTH>

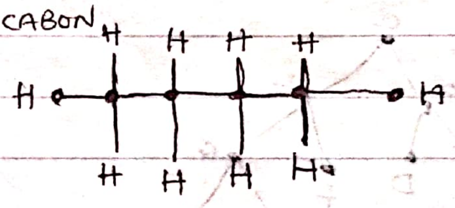
The sum of the level of all pendant vertices of a binary tree is called the Path length <external Path length>

Defn <WEIGHT PATH LENGTH>

If each pendant vertex VI of a binary tree is associated with a positive real number. Then it represents the level VI with a value LI called The weight Path length of the binary tree

SATURATED HYDROCABON





Butane

$C_K H_{2K+2}$

$C_4 H_{12}$

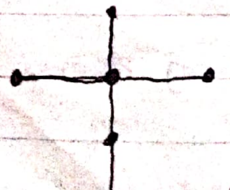
Theorem 27:

The number of labelled trees with n-vertices is n^{n-2} <n ≥ 2>

Defn

A tree (t) is said to be a spanning tree of a connected graph G if t is a subgraph of G and t contains all vertices of G

Graph G



Spanning tree T

<Subgraph of G>

Defn <BRANCH>

An edge which is included in a spanning tree is called a Branch

Defn; <CHORD>

Remaining edges ~~from~~ of the main graph are called Chord.

Theorem 28:

Every connected graph G has atleast one spanning tree.

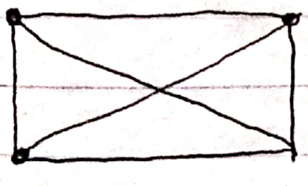
Proof:

If G is a connected graph, then G may or may not be circuitless then it is possible to remove some edges from G such that the graph left is still connected. And this process is continued, that is everytime we remove an edge from a circuit (if any). When this process stop we will have a circuitless connected subgraph of G that contains all vertices of G i.e a spanning tree of G

Defn <PLANAR GRAPH>

A planar graph is a connected graph that can be represented in a plane sphere shape. such that none of the edges cross

Eg: A planar representation of the graph



is